ON SUBSERIES CONVERGENCE IN F-SPACES

BY

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ABSTRACT

In this note, we show that a series, Σx_n, in a complete linear metric space with a basis is subseries convergent if and only if it is weakly subseries convergent.

It is well known that a series, Σx_n, in a B-space is subseries convergent if and only if it is weakly subseries convergent. This theorem was proved by Orlicz in [3] for weakly sequentially complete B-spaces and was proved in the above form by Pettis in [4]. It seems both interesting and useful to know in what class of spaces the above theorem is valid, and more recently Grothendieck [1] and McArthur [2] have shown that the theorem is true in locally convex Hausdorff spaces. Although the theorem clearly cannot be extended to all F-spaces (L_p, $0 < p < 1$, for example) we will prove that it can be extended to all F-spaces with a basis. Our proof will give an alternate method for proving the Orlicz-Pettis theorem in those B-paces whose separable subspaces contain a basis. We begin by stating a lemma whose proof is given in [5].

LEMMA. *Let X be an F-space with a basis, $\{b_n\}$, whose topology is given by the norm p. If for each $x = \Sigma\alpha_j b_j$ in X,*

$$\| x \| = \sup_{m \leq n} p\left(\sum_{j=m}^{n} \alpha_j b_j\right),$$

then $\| \quad \|$ is a (perhaps nonhomogenous) norm on X equivalent to p.

THEOREM. *If X is an F-space with a basis, then a series, Σx_j, in X is subseries convergent if and only if it is weakly subseries convergent.*

PROOF. Let $\{b_n\}$ be a basis for X and let

Received September 17, 1969 and in revised form November 17, 1969

$$\| x \| = \sup_{m \leq n} p\left(\sum_{j=m}^{n} \alpha_j b_j \right)$$

when $x = \Sigma \alpha_j b_j$ and p is any norm giving the topology for X. Let $x_n = \Sigma \alpha_j^n b_j$ and assume that Σx_n is weakly subseries convergent. We show first that $\lim \| x_n \| = 0$. Suppose that this is not true. Then there exists some $\varepsilon > 0$ and a subsequence $\{x_{n_j}\}$ such that $\| x_{n_j} \| > 2\varepsilon$. We suppose for convenience that $\| x_n \| > 2\varepsilon$ for all n. Since $\Sigma f(x_n)$ is subseries convergent, we have $\Sigma |f(x_n)| < \infty$ for each continuous linear functional f. In particular, this holds when f is replaced by a coefficient functional f_n.

Let s_0, $n_1 = 1$, and choose s_1 such that $\| \Sigma_{j=s_1}^{\infty} \alpha_j^{n_1} b_j \| < (\varepsilon/4)$. Then choose $r_1 > 0$ such that $\sup_{1 \leq j \leq s_1} |\alpha_j| < r_1$ implies that $\| \Sigma_{j=1}^{s_1} \alpha_j b_j \| < (\varepsilon/2)$. Having chosen n_k, s_k, and r_k, choose n_{k+1}, s_{k+1}, and r_{k+1} inductively in the following way Choose n_{k+1} such that

$$\sup_{s_{n-1} \leq j \leq s_n} |\alpha_j^{n_{k+1}}| < \frac{r_n}{2^{k-n+1}} \qquad \text{for } n = 1, 2, \cdots, k$$

and such that

$$\left\| \sum_{j=1}^{s_k} \alpha_j^{n_{k+1}} b_j \right\| < \frac{\varepsilon}{2}.$$

Now choose s_{k+1} such that

$$\sum_{i=1}^{k+1} \left\| \sum_{j=s_{k+1}}^{\infty} \alpha_j^{n_i} b_j \right\| < \frac{\varepsilon}{2^{k+2}}.$$

Finally, choose r_{k+1} such that

$$\sup_{s_k \leq j \leq s_{k+1}} |\alpha_j| < r_{k+1}$$

implies

$$\left\| \sum_{j=s_k}^{s_{k+1}} \alpha_j b_j \right\| < \frac{\varepsilon}{2}.$$

Suppose that Σx_{n_i} converges weakly to x and $x = \Sigma \alpha_j b_j$. Since each f_j is continuous, $\alpha_j = \Sigma_{i=1}^{\infty} \alpha_j^{n_i}$

Furthermore, the conditions imposed above imply that

$$\left\| \sum_{j=s_{k-1}}^{s_k} \alpha_j{}^{n_k} b_j \right\| \geq \frac{5}{4}\varepsilon,$$

$$\left\| \sum_{i=1}^{k-1} \sum_{j=s_{k-1}}^{s_k} \alpha_j^{n_i} b_j \right\| < \frac{\varepsilon}{2},$$

and

$$\left\| \sum_{i=k+1}^{\infty} \sum_{j=s_{k-1}}^{s_k} \alpha_j^{n_i} b_j \right\| < \frac{\varepsilon}{2}$$

for all integers k, $k \geqq 1$. Hence, it is easy to see that

$$\left\| \sum_{j=s_k}^{s_{k+1}} \alpha_j b_j \right\| > \frac{\varepsilon}{4} \text{ for } k = 1, 2, \cdots.$$

Thus, $\Sigma \alpha_j b_j$ is not convergent.

We now note that the set of all finite sums of elements of the set $\{x_j\}$ is a norm bounded set. Indeed, if this were not true it would be possible to construct a series, Σy_j, with

$$y_j = \sum_{k=p_i}^{q_i} x_{n_k}$$

where $\{p_j\}$ and $\{q_j\}$ are increasing sequences of positive integers chosen so that $\| y_j \| > \rho > 0$ for all j and Σx_{n_k} is a subseries of Σx_j. Since Σx_j is weakly subseries convergent, Σy_j is weakly subseries convergent and this implies that $\lim \| y_j \| = 0$, a contradiction.

Let

$$M_1 = \sup_{\sigma} \left\| \sum_{i \in \sigma} x_i \right\|$$

where σ is a finite set of positive integers, and let σ_1 be any choice of σ for which $\| \Sigma_{i \in \sigma_1} x_i \| > M_1 - \frac{1}{2}$. Having chosen M_k and σ_k, choose M_{k+1} such that

$$M_{k+1} = \sup_{\sigma} \left\| \sum_{i \in \sigma} x_i \right\|$$

where σ ranges over all finite sets of positive integers such that $\sigma \cap (\bigcup_{i=1}^{k} \sigma_i) = \emptyset$. Now let σ_{k+1} be any admissable σ for which

$$\left\| \sum_{i \in \sigma_{k+1}} x_i \right\| > M_{k+1} - \frac{1}{2^{k+1}}.$$

If $\lim M_k \neq 0$, one can construct, as above, a series Σy_j which is weakly subseries convergent but $\lim \| y_n \| \neq 0$. This is again impossible. Hence $\lim M_k = 0$, and this is easily seen to imply that Σx_j is unordered Cauchy. Since the space, X, is complete Σx_j converges.

In closing, we note that the referee has pointed out that it is apparently not known whether Orlicz's theorem holds in all F-spaces whose weak topology separates points.

References

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